G-convergence of linear self-adjoint operators:

Let us consider the following situation;

$$X := L^2(\Omega), \quad Y := H_0^1(\Omega), \quad \Omega \subseteq \mathbb{R}^N$$ open & bounded regular set

let;

$$A_m u := \sum_{i,j=1}^{N} D_i\left(a_{ij}^m D_j u\right),$$

$$A_m : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega),$$

where:

- $a_{ij}^m \in L^\infty(\Omega)$, $a_{ij}^m = a_{ji}^m$ a.e. in $\Omega$
- $\alpha |\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}^m(x)\, \xi_i \xi_j \leq \beta |\xi|^2$, $0<\alpha \leq \beta<\infty$
  for a.e. $x \in \Omega$, $\forall \xi \in \mathbb{R}^N$.

Assume $a_{ij}^m \longrightarrow a_{ij}$ a.e. in $\Omega$.
Fix $g \in H^{-1}(\Omega)$ and let $u_k, u \in H_0^1(\Omega)$ be the solutions of

$$A_m u_m = g, \qquad A u = g \qquad \text{in } H^{-1}(\Omega).$$

respectively.
Then;

$$\boxed{u_m \xrightarrow{S\text{-}H_0^1} u.}$$

Proof;

i) $\underline{u_k \xrightarrow{w\text{-}H_0^1} u}$: let $\varphi \in H_0^1(\Omega)$; then

(*) $\langle g, \varphi \rangle = \langle A_m u_m, \varphi \rangle = \int_\Omega \sum_{i,j=1}^N a_{ij}^m D_j u_m D_i \varphi \, dx.$

By taking $\varphi = u_m$, we get

$$\alpha \|u_m\|^2_{H_0^1} \leq \langle A u_m, u_m \rangle = \langle g, u_m \rangle \leq \|g\|_{H^{-1}} \|u_m\|_{H_0^1}$$

$\Rightarrow$ $\|u_m\|_{H_0^1}$ uniformly bounded $\underset{\substack{\text{up to a}\\ \text{subsequence}}}{\Longrightarrow}$ $u_m \xrightarrow{w\text{-}H_0^1} v$.

We want to prove that $v = u$.

For, notice that we have:

- $D_j u_m \xrightarrow{w\text{-}L^2} D_j v$
- $a_{ij}^m D_i \varphi \xrightarrow{s\text{-}L^2} a_{ij} D_i \varphi \quad \forall i,j = 1, \dots, N$

[by pointwise convergence + $|\sum_{i=1}^N a_{ij}^m D_i \varphi| \leq B |D_i \varphi|$]

So, we can pass to the limit in (*), getting:

$$\langle g, \varphi \rangle = \int_\Omega \sum_{i,j=1}^N a_{ij} D_j u_m D_i \varphi \, dx = \langle A u, \varphi \rangle.$$

So $u = v$.

ii) $u_m \xrightarrow{s-H_0^1} u$:

$$\alpha \| u - u_m \|^2_{H_0^1} \leq \langle A_m (u_m - u), u_m - u \rangle$$

$$= \langle A_m u_m, u_m \rangle - 2 \langle A_m u_m, u \rangle + \langle A_m u, u \rangle$$

$$= \langle g, u_m \rangle - 2 \langle g, u \rangle + \boxed{\langle A_m u, u \rangle}$$

$$\to \langle g, u \rangle - 2 \langle g, u \rangle + \boxed{\langle A u, u \rangle}$$

$$= 0$$

by pointwise convergence of $a_{ij}^m \to a_{ij}$

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$\to$ We would like to understand whether it is possible to generalize the above situation to the case where the operators $A_m$ converge in a weaker sense to $A$, but in such a way that the solutions of the problems $A_m u_m = g$ still converge to the solution of $Au = g$.

The setting we want to consider is the following:

let $X, Y$ be separable Hilbert spaces, $Y$ compactly contained in $X$, and dense in $X$.

For $0 < \alpha \leq \beta < +\infty$, set

$$\mathcal{A}_{\alpha,\beta} := \{ A : Y \to Y' ; \ A \text{ positive, self-adjoint}, \ \alpha \|u\|_Y^2 \leq \langle Au, u \rangle \leq \beta \|u\|_Y^2 \}$$

· **Def:** we say that a sequence $(A_m)_m \subset \mathcal{A}_{\alpha,\beta}$ <u>G-converges strongly</u> [<u>weakly</u>] to $A \in \mathcal{A}_{\alpha,\beta}$ if

$$A_m^{-1} g \xrightarrow{s-Y} A^{-1} g \qquad [\, A_m^{-1} g \xrightarrow{w-Y} A^{-1} g \,]$$

for all $g \in Y$!

· Notice that

$$Au = g$$

is the Euler-Lagrange equation of the functional

$$L(u) := \langle Au, u \rangle - 2 \langle g, u \rangle,$$

and that $F(u) := \langle Au, u \rangle$ is a <u>quadratic form</u> on $Y$.
Moreover, $L$ <u>admits a unique minimum point</u> in $Y$:
indeed:

· $L$ strictly convex $[\alpha > 0]$
· $L$ w-l.s.c. [
· $L$ seq-coercive $[\lim_{\|x\| \to +\infty} L(x) = +\infty]$

→ The connection between quadratic forms and self-adjoint positive operators is clear:

$$F \in Q_{\alpha,\beta} \xleftrightarrow{F(u) = \langle Au, u \rangle} A \in \mathcal{A}_{\alpha,\beta},$$

where:

$$Q_{\alpha,\beta} := \{ F : Y \to [0, \infty) : F \text{ quadratic}, \ \alpha \|x\|^2 \le F(x) \le \beta \|x\|^2 \}.$$

It is easy to see that;

$$(F_n)_n \in Q_{\alpha,\beta}\ ,\ F_n \xrightarrow{\Gamma} F \text{ [in the strong or in the weak topology]}$$

$$\Downarrow$$

$$F \in Q_{\alpha,\beta}$$

[this is done by using the fact that:

$F : Y \to [0,\infty)$ s.t.

i) $F(0)=0$

ii) $F(tx) \le t^2 F(x) \quad \forall t>0\ \forall x \in Y$ $\Rightarrow$ F quadratic]

iii) $F(x+y)+F(x-y) \le 2(F(x)+F(y))$
$\forall x,y \in Y$

Moreover, by using the integral representation of $\Gamma$-limits, it is possible to prove the following:

let $(a_{ij})_{i,j=1}^{N}$ be $L^\infty(\Omega)$ functions s.t.

- $a_{ij} = a_{ji}$ a.e. in $\Omega$
- $\alpha |\xi|^2 \le \sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \le \beta |\xi|^2 \quad \forall \xi \in \mathbb{R}^N$

and consider the class $\tilde{Q}_{\alpha,\beta}$ of all quadratic functionals $F : L^2(\Omega) \to [0,\infty]$ of the form:

$$F(u) = \begin{cases} \int_\Omega \left(\sum_{i,j=1}^{N} a_{ij} D_i u D_j u\right) dx & u \in H^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Then, $\forall (F_n)_n \subset \tilde{Q}_{\alpha,\beta}$ there exists a subsequence $(F_{n_k})_k$ and $F \in \tilde{Q}_{\alpha,\beta}$ s.t. $F_{n_k} \xrightarrow{\Gamma-L^2} F$.

In particular, if $F_n \xrightarrow{\Gamma-L^2} F$, then $F \in \tilde{Q}_{\alpha,\beta}$.

- The relation between $G$-convergence & $\Gamma$-convergence is given by the following result:

  - <u>Thm</u>:

    Let $(F_n)_n \subset Q_{\alpha,\beta}$ and let $(A_n)_n \subset \mathcal{A}_{\alpha,\beta}$ be the corresponding operators.

    Then, the following are equivalent:

    i) $F_n \xrightarrow{\Gamma-w_Y} F$

    ii) $\min\limits_{y \in Y} [F_n(y) - 2\langle g, y\rangle] \to \min\limits_{y\in Y} [F(y) - 2\langle g, y\rangle]$
    for all $g \in Y'$

    iii) $A_n \xrightarrow{G-w_Y} A$

<u>Proof</u>:

- i) $\Rightarrow$ ii): $(F_n)_n$ is $w$-equi-coercive
- ii) $\Rightarrow$ iii): let $u_n \in Y$ be the minimum of

$$L_n(u) := F_n(u) - 2\langle g, u\rangle.$$

Then $u_n$ satisfies the EL-equation:

$$A_n u_n = g.$$

In particular, we have that:

$$F_m(u_m) - 2\langle g, u_m\rangle = \langle A_m u_m, u_m\rangle - 2\langle g, u_m\rangle$$

$$= -\langle g, u_m\rangle =$$

$$= -\langle g, A_m^{-1} g\rangle$$

By hypothesis, the left-hand side converges to

$$F(u) - 2\langle g, u\rangle = -\langle g, A^{-1} g\rangle$$

$u$ is the minimum of $F(y) - 2\langle g, y\rangle$ in $Y$

Thus, we get,

$$(*) \quad \langle g, A_m^{-1} g\rangle \longrightarrow \langle g, A^{-1} g\rangle$$

for all $g \in Y'$!

What we have to prove is that $A_m^{-1} g \xrightarrow{w\text{-}Y} A^{-1} g$, for all $g \in Y'$. So, fix $g, f \in Y'$, and write:

$$\langle f, A_m^{-1} g\rangle = \frac{1}{4}\left[\langle f+g, A_m^{-1}(f+g)\rangle - \langle f-g, A_m^{-1}(f-g)\rangle\right]$$

$$\xrightarrow{\text{by } (*)} \frac{1}{4}\left[\langle f+g, A^{-1}(f+g)\rangle - \langle f-g, A^{-1}(f-g)\rangle\right]$$

$$= \langle f, A^{-1} g\rangle.$$

So we get our result.

• iii) $\Rightarrow$ i) : because of our hypothesis, $\Gamma$-convergence in the weak topology is characterized by the two sequential properties.
So:

• <u>$\Gamma$-limsup</u>: let $u \in Y$ and consider the problem:

$$\begin{cases} A_n u_n = Au, \\ u_n \in Y. \end{cases}$$

By iii) we get that: $u_n \xrightarrow{w-Y} u$.
Moreover, we have that:

$$F_n(u_n) = \langle A_n u_n, u_n \rangle \underset{\substack{\downarrow \\ \text{by the equation}}}{=} \langle Au, u_n \rangle \longrightarrow \langle Au, u \rangle = F(u)$$

• <u>$\Gamma$-liminf</u>: let $u \in Y$ and $v_n \xrightarrow{w-Y} u$. Then:

$$F_n(v_n) = F_n(-u_n + v_n + u_n)$$

$$= \underbrace{F_n(v_n - u_n)}_{\geq 0} + 2\langle \underbrace{A_n u_n}_{= Au}, v_n - u_n \rangle + F_n(u_n)$$

$$\geq \underbrace{2\langle Au, v_n - u_n \rangle}_{\to 0} + \underbrace{F_n(u_n)}_{\to F(u)}.$$

- Remark: the equivalence between i) and ii) is peculiar of quadratic functionals; it says that, in order to have $F_n \xrightarrow{\Gamma-u} F$, we just need to check that the minimum of every linear perturbation of $F_n$ converges to the minimum of $F$ perturbed in the same way.

In general, for a metric space [or in a situation like the one we have] it holds that:

let $(F_n)_n : X \to [0,\infty]$ equi-coercive, and let $F : X \to [0,\infty]$ l.s.c.. Then:

$$F_n \xrightarrow{\Gamma} F \iff \text{for every } G: X \to [0,\infty) \text{ } \underline{\text{continuous}} \quad \inf_X (F+G) = \lim_n \inf_X (F_n+G).$$

- The quadratic forms $F_n$ are defined only on $Y \subset X$. With an abuse of notation, it is possible to extend them to the whole space $X$ as follows:

$$F_n(x) := \begin{cases} F_n(x) & \text{if } x \in Y, \\ +\infty & \text{otherwise in } X \setminus Y. \end{cases}$$

We do the same with $F$:

$$F(x) := \begin{cases} F(x) & \text{if } x \in Y, \\ +\infty & \text{in } X \setminus Y. \end{cases}$$

We have the following:

- <u>Thm</u>:

$$F_n \xrightarrow{\Gamma - s_X} F \iff F_{n|Y} \xrightarrow{\Gamma - w_Y} F_{|Y}.$$

<u>Proof</u>:

- $\Rightarrow$: - <u>$\Gamma$-limsup</u>: let $u \in X$; then $\exists \boxed{u_n \xrightarrow{s-X} u}$ s.t. $F(u) = \lim_n F_n(u_n)$.

Let $u \in Y \Rightarrow F(u) < +\infty \Rightarrow F_n(u_n)$ bounded.

But: $\alpha \|u_n\|_Y^2 \leq F_n(u_n) \leq M$

$\Rightarrow u_n \xrightarrow{w-Y} v \underset{\text{compact embedding}}{\Rightarrow} u_{n_k} \xrightarrow{s-X} v \Rightarrow v = u$

Since this can be done for every subsequence, we get $u_n \xrightarrow{w-Y} u$.

- <u>$\Gamma$-liminf</u>: let $u \in Y$, $u_n \xrightarrow{w-Y} u \underset{\text{compact embedding}}{\Rightarrow} u_n \xrightarrow{s-X} u$.

Thus: $F(u) \leq \liminf_n F_n(u_n)$.

- $\Leftarrow$: - <u>$\Gamma$-liminf</u>: let $u \in X$ and $u_n \xrightarrow{s-X} u$;

assume $\liminf_n F_n(u_n) = \lim_n F_n(u_n) < +\infty$.

So, $u_n \in Y$ and $(u_n)_n$ is bounded

$\Rightarrow u_{n_k} \xrightarrow{w-Y} u$.

So:

$$F(u) \leq \liminf_k F_{n_k}(u_{n_k}) = \lim_k F_{n_k}(u_{n_k}) =$$

$$= \liminf_n F_n(u_n).$$

• $\Gamma$-limsup: let $u \in X$ s.t. $F(u) < +\infty \Rightarrow u \in Y$

$$\Rightarrow \exists\, u_n \xrightarrow{w\text{-}Y} u \text{ s.t. } F(u) = \lim_n F_n(u_n)$$

$$\underset{\text{compact embedding}}{\Rightarrow} \quad u_n \xrightarrow{s\text{-}X} u$$

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• Similarly, it is possible to prove that:

$$F_n \xrightarrow{\Gamma\text{-}s_X} F \iff F_n \xrightarrow{\Gamma\text{-}w_X} F.$$

Moreover, it also holds that:

$$\begin{cases} F_n \xrightarrow{\Gamma\text{-}s_X} F \\ F_n \xrightarrow{\Gamma\text{-}w_X} F \end{cases} \iff A_n^{-1} g \xrightarrow{s\text{-}X} A^{-1} g \quad \forall g \in X'.$$

The bottom line is that the following are equivalent:

i) $F_m \xrightarrow{\Gamma - s_X} F$

ii) $F_m|_Y \xrightarrow{\Gamma - w_Y} F|_Y$

iii) $\forall g \in X'$; $\min_{u \in X} [F_m(u) - 2\langle g, u\rangle] \to \min_{u \in X} [F(u) - 2\langle g, u\rangle]$

iv) $\forall g \in Y'$: $\min_{u \in Y} [F_m(u) - 2\langle g, u\rangle] \to \min_{u \in Y} [F(u) - 2\langle g, u\rangle]$

v) $\forall g \in X'$, the solution $u_m$ of

$$\begin{cases} A_m u_m = g, \\ u_m \in Y, \end{cases}$$

converges <u>strongly</u> to the solution $u$ of

vi) $\forall g \in Y'$, the solution $u_m$ of

$$\begin{cases} A_m u_m = g, \\ u_m \in Y, \end{cases}$$

converges <u>weakly</u> to the solution $u$ of

$$\begin{cases} Au = g, \\ u \in Y. \end{cases}$$

Let $(A_n)_n \xrightarrow{\sigma-w} A$, and let $(\lambda_i^n)_i$, $(\lambda_i)_i$ be the of eigenvalues.

Then:

i) $\lambda_i^n \to \lambda_i$

ii) if $\lambda_{i-1} < \lambda_i = \lambda_{i+1} = \dots = \lambda_{i+m_i} < \lambda_{i+m_i+1}$, let $E(u_i, \dots, u_{i+m_i})$ be the eigenspace of $\lambda_i, \dots, \lambda_{i+m_i}$, and let $E_n(u_i^n, \dots, u_{i+m_i}^n)$ be the space generated by $u_i^n, \dots, u_{i+m_i}^n$.
Then:

$$E_n(u_i^n, \dots, u_{i+m_i}^n) \xrightarrow{K} E(u_i, \dots, u_{i+m_i}),$$

iii) if $\lambda$ is <u>not</u> an eigenvalue of $A$, then

$$\forall g \in X' \quad (A_n - \lambda Id)^{-1} g \xrightarrow[w-Y]{s-X} (A - \lambda Id)^{-1} g.$$

- Some variants;

i) $g_m \xrightarrow{s-Y'} g$, $A_m \xrightarrow{G-\nu} A$, then the solution $u_m$ of the problem

$$\begin{cases} A_m u_m = g_m \\ u_m \in Y \end{cases}$$

weakly converges in $Y$ to the solution of

$$\begin{cases} Au = g \\ u \in Y \end{cases}$$

ii) Let $(A_m)_m$ be a sequence of operators in $\mathcal{A}_{\alpha,\beta}$ in divergence form that $G$-converges in $L^2$ to an operator $A$ [ $A$ will belong to $\mathcal{A}_{\alpha,\beta}$ and will be in divergence form.

Let $u_m$ be the solution of

$$\begin{cases} A_m u_m = g & \text{in } \Omega, \\ u_m - \varphi \in H_0^1, \end{cases}$$

where $g \in L^2$, $\varphi \in H_0^1$. Then $u_m \xrightarrow{L^2} u$, where $u$ solves

$$\begin{cases} Au = g & \text{in } \Omega \\ u - \varphi \in H_0^1 \end{cases}$$

iii) In the same hypothesis, assume also $\partial\Omega$ Lipschitz, $\lambda > 0$ and let $u_m$ be the unique weak solution to

$$\begin{cases} A_m u_m + \lambda u_m = g & \text{in } \Omega, \\ \sum_{i,j=1}^{N} a_{ij}^m D_j u_m \nu_i = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $u_m \xrightarrow{L^2} u$, where $u$ solves

$$\begin{cases} Au + \lambda u = g & \text{in } \Omega, \\ \sum_{i,j=1}^{N} a_{ij} D_j u \nu_i = 0 & \text{on } \partial\Omega. \end{cases}$$